

# Existence of CMC and constant areal time foliations in $T^2$ symmetric spacetimes with Vlasov matter

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## Abstract

The global structure of solutions of the Einstein equations coupled to the Vlasov equation is investigated in the presence of a two-dimensional symmetry group. It is shown that there exist global CMC and areal time foliations. The proof is based on long-time existence theorems for the partial differential equations resulting from the Einstein-Vlasov system when conformal or areal coordinates are introduced.

## 1 Introduction

In general relativity the gravitational field is expressed in terms of the geometry of spacetime. The matter constituting self-gravitating physical systems is described using certain matter fields which are also geometrical objects. The spacetime metric is required to satisfy the Einstein equations and these are coupled to equations of motion for the matter fields. The resulting

Einstein-matter system of partial differential equations is the central mathematical element of the theory. Thus to understand the mathematical content of general relativity we require an overview of the solutions of these equations and their qualitative behaviour.

When different kinds of physical situations are considered solutions of the Einstein-matter equations with certain boundary conditions or spatial asymptotics will be particularly relevant. One choice which avoids the issue of boundary conditions or asymptotic conditions in the strict sense is to give initial data on a compact manifold. This possibility, which is that studied in the present paper, is of interest for applications to cosmology. In practice it leads to imposing periodic boundary conditions.

A further choice which has to be made in order to get a concrete mathematical problem is to decide which kind of matter fields to consider. Here we choose collisionless matter, where the equation of motion is the Vlasov equation. The special interest of this kind of matter has been discussed in several places (see [R13], [R15], [A2]) and the relevant facts will not be repeated here. A fundamental insight is that collisionless matter acts as a source of the gravitational field in a way which allows specific features of the matter model to remain in the background while revealing basic properties of the dynamics of self-gravitating matter. A fact on the PDE level related to this is that in many cases long-time existence theorems for solutions of the Einstein-Vlasov system are to be expected. For other matter models coupled to the Einstein equations this is not true. Results on formation of singularities in finite time in the case of one matter model, dust, can be found in [R14] and [I].

While the Einstein-matter system itself is naturally expressed in geometrical terms the application of the theory of partial differential equations to study it requires the geometry to be parametrized in a suitable way by the use of coordinates or other auxiliary constructs. An important step towards investigating the dynamics of solutions is to find parametrizations which are at once practical and applicable in sufficient generality. Of particular significance is finding a function which can act as a good time coordinate in the situation of interest. This task is the focus of the following. It will be shown that under appropriate symmetry assumptions it can be solved in a very satisfactory way.

A central role in solving the geometric problems addressed in this paper is played by two long-time existence results for certain systems of PDE in one space dimension. One of these is a global in time existence theorem while the other is a continuation criterion which says that solutions continue to exist as long as a certain quantity does not vanish. A number of techniques

used in the proofs are adapted from known arguments for the Vlasov-Poisson and Vlasov-Maxwell systems while others are specific to the Einstein-Vlasov case.

Consider a solution of the Einstein-Vlasov system evolving from initial data on a compact spacelike hypersurface. This paper is mainly concerned with the case where the initial hypersurface is a three-dimensional torus with periodic coordinates  $(\theta, x, y)$ . Moreover solutions are considered where both the metric and the matter fields are invariant under translations in the  $x$  and  $y$  directions. Taking account of the periodic identifications involved we see that these solutions admit a two-dimensional symmetry group isomorphic to a two-torus  $T^2 = S^1 \times S^1$ . These spacetimes will be referred to as  $T^2$  symmetric. No additional symmetry assumptions will be imposed. Analogues of the results of this paper for various cases with higher symmetry, including certain subcases of  $T^2$  symmetry, have been obtained previously. A survey of these earlier results can be found in the introduction of [ARR]. The present paper can be seen as the culmination of a development concerning global geometrically defined time coordinates in solutions of the Einstein-Vlasov system with at least two symmetries.

On a  $T^2$  symmetric spacetime it is possible to define a function  $R$ , the area function, as follows. If  $p$  is a point of the spacetime then  $R(p)$  is equal to the area of the orbit of the action of  $T^2$  which contains  $p$ . Evidently the function  $R$  is itself invariant under the action of  $T^2$ . We say that a  $T^2$  symmetric initial data set is *flat* if the spacetime gradient of  $R$  in a Cauchy development of this data set vanishes identically on the initial hypersurface. Otherwise it will be called non-flat. Whether an initial data set is flat in this sense can be determined intrinsically from the initial data without having to know anything about a Cauchy development. The reason for the terminology is that, as follows from [R11], if an initial data set is flat in this sense any Cauchy development of it is flat in the sense that its Riemann curvature tensor vanishes everywhere. Cf. the discussion in Section 3.

The main theorems will now be stated. The first concerns the existence of a global time coordinate of constant mean curvature. Consider a spacetime which evolves from initial data on a compact hypersurface. In the following it will always be assumed implicitly that the initial datum for the particle density has compact support. A real-valued function  $t$  on the spacetime is called a constant mean curvature (CMC) time coordinate if each of its level hypersurfaces is compact and spacelike, it has constant mean curvature and the value of the mean curvature there is equal to  $t$ .

**Theorem 1** *Let  $(M, g_{\alpha\beta}, f)$  be the maximal globally hyperbolic development*

of non-flat  $C^\infty$  initial data for the Einstein-Vlasov system with  $T^2$  symmetry. Then  $M$  can be covered by compact spacelike hypersurfaces of constant mean curvature with each value in the range  $(-\infty, 0)$  occurring as the mean curvature of precisely one of these hypersurfaces.

The second theorem concerns the existence of an areal time coordinate. Consider a spacetime with  $T^2$  symmetry. A real-valued function  $t$  on the spacetime is called an areal time coordinate if each of its level hypersurfaces is compact and spacelike and the value of  $t$  on the hypersurface is everywhere equal to that of the area function  $R$ .

**Theorem 2** *Let  $(M, g_{\alpha\beta}, f)$  be the maximal globally hyperbolic development of non-flat  $C^\infty$  initial data for the Einstein-Vlasov system with  $T^2$  symmetry. Then  $M$  can be covered by compact spacelike hypersurfaces of constant area function  $R$  with each value in the range  $(R_0, \infty)$  occurring as the value of the area function on precisely one of these hypersurfaces. Here  $R_0$  is a non-negative real number.*

Notice that these theorems include the vacuum case  $f = 0$ . Theorem 2 was proved in the vacuum case in [BCIM]. Under the additional assumption of Gowdy symmetry, which consists in augmenting the  $T^2$  action by a suitable reflection symmetry, Theorem 2 was proved in [A1]. In [ARR] an argument was sketched which indicates that Theorem 1 also holds under the assumption of Gowdy symmetry. The results of both theorems extend to the case of local  $U(1) \times U(1)$  symmetry, as defined in [R11]. Since there is no essential difference in the proofs this generalization will not be mentioned further. The fact that the theorems are stated for  $C^\infty$  initial data is a matter of convenience. Keeping track of derivatives would allow analogous results to be proved for initial data of finite differentiability.

The paper is structured as follows. The next section contains some definitions which will be needed. The two sections after that contain the basic PDE analysis in the contracting and expanding directions respectively. The proofs of the main theorems are given in Section 5.

## 2 The Einstein-Vlasov system with $T^2$ symmetry

Consider the manifold  $M = \mathbb{R} \times T^3$ . Let  $T^2$  act on  $T^3$  in the obvious way, arising from the action of  $\mathbb{R}^2$  on  $\mathbb{R}^3$  with Cartesian coordinates  $(\theta, x, y)$  by translations in  $x$  and  $y$ . Correspondingly  $T^2$  acts on  $M$  with coordinates  $(t, \theta, x, y)$ . A spacetime with underlying manifold  $M$  defined by a metric

$g_{\alpha\beta}$  and matter fields is said to be  $T^2$  symmetric if the metric and matter fields are invariant under the action of  $T^2$ . The orbits of the group action will be referred to as surfaces of symmetry and a hypersurface will be called symmetric if it is a union of surfaces of symmetry. It is now clear how to define abstract Cauchy data for the Einstein-matter equations with  $T^2$  symmetry. They should be defined on  $T^3$  and invariant under the action of  $T^2$ . It will be assumed throughout that the metric and the matter fields are  $C^\infty$ .

Suppose now that a matter model is chosen for which the Cauchy problem for the Einstein equations is well-posed. The example of interest in the following is that of collisionless matter satisfying the Vlasov equation. Corresponding to initial data for the Einstein-matter equations with  $T^2$  symmetry there is a maximal Cauchy development. We will construct a certain local coordinate system on a neighbourhood of the initial hypersurface in the maximal Cauchy development. On the initial hypersurface itself we choose periodic coordinates  $(\theta, x, y)$  as above. The isometries are given by translations in  $x$  and  $y$ . These coordinates can be extended uniquely to a Gaussian coordinate system  $(t, \theta, x, y)$  on a neighbourhood of the initial hypersurface. On general grounds the action of  $T^2$  on the initial data extends uniquely to an action on the maximal Cauchy development  $M$  by symmetries (see e.g. [FR], Section 5.6). Gauss coordinates inherit the symmetries of their initial values and the spacetime. Hence the components of the metric in these coordinates depend only on  $t$  and  $\theta$ . It is then a matter of simple algebra to see that in these coordinates the metric can be written in the form

$$-dt^2 + e^{2(\hat{\eta}-U)}d\theta^2 + e^{2U}[dx + Ady + (G + AH)d\theta]^2 + e^{-2U}R^2[dy + Hd\theta]^2. \quad (1)$$

for functions  $(\hat{\eta}, U, A, G, H, R)$  of  $t$  and  $\theta$  which are periodic in  $\theta$ . It can also be read off that the function  $R$  in this form of the metric coincides with the area function mentioned in the introduction.

A region covered by coordinates of this type which is invariant under the action of  $T^2$  can be quotiented by the group action to get a two dimensional quotient manifold  $Q$  coordinatized by  $t$  and  $\theta$ . The quotient inherits a Lorentzian metric by the rule that the inner product of two vectors on the quotient is equal to the inner product of the unique vectors on spacetime which project onto them orthogonally to the orbit. On  $Q$  we can pass to double null coordinates  $(u, v)$  on a neighbourhood of the quotient of the initial hypersurface, i.e. to coordinates whose level curves are null. Defining new coordinates by  $t = \frac{1}{2}(u - v)$  and  $\theta = \frac{1}{2}(u + v)$  puts the metric on  $Q$  into conformally flat form. By pull-back these define new coordinates on  $M$

where the metric takes the form

$$g = e^{2(\eta-U)}(-dt^2 + d\theta^2) + e^{2U}[dx + A dy + (G + AH)d\theta]^2 + e^{-2U}R^2[dy + Hd\theta]^2. \quad (2)$$

for functions  $(\eta, U, A, G, H, R)$  of  $t$  and  $\theta$ . A coordinate system of this type is called a conformal coordinate system. It has now been seen that conformal coordinates always exist on some neighbourhood of the initial hypersurface in a spacetime evolving from data with  $T^2$  symmetry. In addition it is possible to choose the double null coordinates in such a way that the initial hypersurface coincides with  $t = 0$  and the metric coefficients are periodic in  $\theta$ .

To conclude this section we formulate the Einstein-Vlasov system which governs the time evolution of a self-gravitating collisionless gas in the context of general relativity; for the moment we do not assume any symmetry of the spacetime. All the particles in the gas are assumed to have the same rest mass, normalized to unity, and to move forward in time so that their number density  $f$  is a non-negative function supported on the mass shell

$$PM := \{ \eta_{\mu\nu} v^\mu v^\nu = -1, v^0 > 0 \},$$

a submanifold of the tangent bundle  $TM$  of the space-time manifold  $M$  with metric  $g_{\alpha\beta}$ . Here  $\eta_{\mu\nu}$  denotes the components of the Minkowski metric and  $v^\mu$  denote the components of a tangent vector in an orthonormal frame  $e_\mu$ . We use coordinates  $(t, x^a)$  with zero shift and the frame components  $v^i$  to parametrize the mass shell; Greek indices always run from 0 to 3, and Latin ones from 1 to 3. On the mass shell  $PM$  the variable  $v^0$  becomes a function of the  $v^i$ :

$$v^0 = \sqrt{1 + \delta_{ij} v^i v^j}.$$

The Einstein-Vlasov system now reads

$$v^\mu e_\mu(f) - \gamma_{\mu\nu}^l v^\mu v^\nu \frac{\partial f}{\partial v^l} = 0$$

$$G^{\mu\nu} = 8\pi T^{\mu\nu},$$

$$T^{\mu\nu} = \int v^\mu v^\nu f \frac{dv^1 dv^2 dv^3}{-v_0}$$

where  $\gamma_{\mu\nu}^\lambda$  are the Ricci rotation coefficients,  $G^{\mu\nu}$  the Einstein tensor, and  $T^{\mu\nu}$  the energy-momentum tensor. In the case of  $T^2$  symmetry the number density  $f$  is assumed to be invariant under the action induced on  $PM$  by

the action of  $T^2$  on  $M$ . The orthonormal frame used to parametrize  $PM$  in the following is also assumed to be invariant. As a result  $f$  is independent of  $x$  and  $y$  and is a function of the variables  $(t, \theta, v^1, v^2, v^3)$ . An explicit choice of invariant orthonormal frame for the metric (2) is given by

$$e^{U-\eta} \frac{\partial}{\partial t}, e^{U-\eta} \left( \frac{\partial}{\partial \theta} - G \frac{\partial}{\partial x} - H \frac{\partial}{\partial y} \right), e^{-U} \frac{\partial}{\partial x}, e^U R^{-1} \left( \frac{\partial}{\partial y} - A \frac{\partial}{\partial x} \right)$$

### 3 Analysis in the contracting direction

As we will see in more detail later, a non-flat  $T^2$  symmetric solution of the Einstein-Vlasov system represents a cosmological model which, after a suitable choice of time orientation, evolves from an initial singularity and has a phase of unlimited expansion at late times. In this section we consider the Einstein-Vlasov system in the contracting direction (i.e. evolving towards the initial singularity) and use conformal coordinates in which the metric takes the form (2). That coordinates of this type can always be found near the initial hypersurface was shown in the last section. The invariant orthonormal frame adapted to these coordinates exhibited there will be used in this section. The explicit form of the Einstein-Vlasov system will now be given. We set  $\Gamma = G_t + AH_t$ . The quantities  $\Gamma$  and  $H_t$  will be referred to as *twist quantities*.

The Einstein-matter constraint equations

$$\begin{aligned} U_t^2 + U_\theta^2 + \frac{e^{4U}}{4R^2}(A_t^2 + A_\theta^2) + \frac{R_{\theta\theta}}{R} - \frac{\eta_t R_t}{R} - \frac{\eta_\theta R_\theta}{R} = \\ = -\frac{e^{-2\eta+4U}}{4}\Gamma^2 - \frac{R^2 e^{-2\eta}}{4}H_t^2 - e^{2(\eta-U)}\rho \end{aligned} \quad (3)$$

$$2U_t U_\theta + \frac{e^{4U}}{2R^2}A_t A_\theta + \frac{R_{t\theta}}{R} - \frac{\eta_t R_\theta}{R} - \frac{\eta_\theta R_t}{R} = e^{2(\eta-U)}J_1 \quad (4)$$

The Einstein-matter evolution equations

$$\begin{aligned} U_{tt} - U_{\theta\theta} &= \frac{U_\theta R_\theta}{R} - \frac{U_t R_t}{R} + \frac{e^{4U}}{2R^2}(A_t^2 - A_\theta^2) + \frac{e^{-2\eta+4U}}{2}\Gamma^2 \\ &\quad + \frac{1}{2}e^{2(\eta-U)}(\rho - P_1 + P_2 - P_3) \\ A_{tt} - A_{\theta\theta} &= \frac{R_t A_t}{R} - \frac{R_\theta A_\theta}{R} + 4(A_\theta U_\theta - A_t U_t) + R^2 e^{-2\eta}\Gamma H_t \end{aligned} \quad (5)$$

$$+2Re^{2(\eta-2U)}S_{23} \quad (6)$$

$$R_{tt} - R_{\theta\theta} = Re^{2(\eta-U)}(\rho - P_1) + \frac{Re^{-2\eta+4U}}{2}\Gamma^2 + \frac{R^3e^{-2\eta}}{2}H_t^2, \quad (7)$$

$$\begin{aligned} \eta_{tt} - \eta_{\theta\theta} = & U_\theta^2 - U_t^2 + \frac{e^{4U}}{4R^2}(A_t^2 - A_\theta^2) - \frac{e^{-2\eta+4U}}{4}\Gamma^2 \\ & - \frac{3R^2e^{-2\eta}}{4}H_t^2 - e^{2(\eta-U)}P_3 \end{aligned} \quad (8)$$

Auxiliary equations

$$\partial_\theta[Re^{-2\eta+4U}\Gamma] = -2Re^\eta J_2, \quad (9)$$

$$\partial_t[Re^{-2\eta+4U}\Gamma] = 2Re^\eta S_{12}, \quad (10)$$

$$\partial_\theta(R^3e^{-2\eta}H_t) + Re^{-2\eta+4U}A_\theta\Gamma = -2R^2e^{\eta-2U}J_3, \quad (11)$$

$$\partial_t(R^3e^{-2\eta}H_t) + Re^{-2\eta+4U}A_t\Gamma = 2R^2e^{\eta-2U}S_{13}. \quad (12)$$

The Vlasov equation

$$\begin{aligned} & \frac{\partial f}{\partial t} + \frac{v^1}{v^0} \frac{\partial f}{\partial \theta} - \left[ (\eta_\theta - U_\theta)v^0 + (\eta_t - U_t)v^1 - U_\theta \frac{(v^2)^2}{v^0} \right. \\ & + (U_\theta - \frac{R_\theta}{R}) \frac{(v^3)^2}{v^0} - \frac{A_\theta}{R} e^{2U} \frac{v^2 v^3}{v^0} + e^{-\eta} (e^{2U} \Gamma v^2 + R H_t v^3) \left. \right] \frac{\partial f}{\partial v^1} \\ & - \left[ U_t v^2 + U_\theta \frac{v^1 v^2}{v^0} \right] \frac{\partial f}{\partial v^2} \\ & - \left[ \left( \frac{R_t}{R} - U_t \right) v^3 - \left( U_\theta - \frac{R_\theta}{R} \right) \frac{v^1 v^3}{v^0} + \frac{e^{2U} v^2}{R} (A_t + A_\theta \frac{v^1}{v^0}) \right] \frac{\partial f}{\partial v^3} = 0. \end{aligned} \quad (13)$$

The matter quantities

$$\rho(t, \theta) = \int_{\mathbb{R}^3} v^0 f(t, \theta, v) dv \quad (14)$$

$$P_k(t, \theta) = \int_{\mathbb{R}^3} \frac{(v^k)^2}{v^0} f(t, \theta, v) dv, \quad k = 1, 2, 3 \quad (15)$$

$$J_k(t, \theta) = \int_{\mathbb{R}^3} v^k f(t, \theta, v) dv \quad (16)$$

$$S_{jk}(t, \theta) = \int_{\mathbb{R}^3} \frac{v^j v^k}{v^0} f(t, \theta, v) dv \quad (17)$$



Let a smooth  $T^2$  symmetric solution of the Einstein-Vlasov system written in conformal coordinates be given on some time interval  $(t_-, t_0]$ . We want to show that if this interval is bounded and if  $R$  is bounded away from zero there then  $f$ ,  $R$ ,  $\eta$ ,  $U$ ,  $A$ ,  $G$ ,  $H$  and all their derivatives are bounded as well, with bounds depending on the data at  $t = t_0$  and the lower bound on  $R$ , and that the supremum of the support of momenta at time  $t$ ,

$$Q(t) := \sup\{|v| : \exists (s, \theta) \in [t, t_0] \times S^1 \text{ such that } f(s, \theta, v) \neq 0\}, \quad (18)$$

is uniformly bounded. Note that these conditions imply that the matter quantities and their derivatives are uniformly bounded. In proving these statements it is assumed that the initial data at  $t = t_0$  are non-flat, as in the assumptions of the main theorems.

The description of the proofs in this section is modelled on that of [A1] and highlights the places where there are differences due to non-vanishing twist.

*Step 1.* (Monotonicity of  $R$  and bounds on its first derivatives.)

This is a key step and follows the arguments in [BCIM]. We have to check that the matter terms have the right signs so that these arguments still hold. The bounds on  $R$  and its first derivatives will play a crucial role when we control the matter terms below.

First we note that  $\nabla R$  is timelike. This is a consequence of Proposition 3.1 of [R11]. A  $T^2$  symmetric solution of the Einstein-Vlasov system satisfies the hypotheses of that proposition. Since the initial data set is non-flat the Hawking mass is non-zero somewhere. Then the proposition implies that it is non-zero everywhere. As a consequence  $\nabla R$  is everywhere timelike.

Next we show that  $\partial_t R$  and  $|\partial_\theta R|$  are bounded into the past. Let us introduce the null vector fields

$$\partial_\xi = \frac{1}{\sqrt{2}}(\partial_t + \partial_\theta), \quad \partial_\lambda = \frac{1}{\sqrt{2}}(\partial_t - \partial_\theta), \quad (19)$$

and let us set  $F_\xi = \partial_\xi F$ ,  $F_\lambda = \partial_\lambda F$  for any function  $F$ . The evolution equation (7) can be written

$$\partial_\lambda R_\xi = \frac{R}{2} e^{2(\eta-U)} (\rho - P_1) + \frac{R e^{-2\eta}}{4} (e^{4U} \Gamma^2 + R^2 H_t^2), \quad (20)$$

or equivalently,

$$\partial_\xi R_\lambda = \frac{R}{2} e^{2(\eta-U)} (\rho - P_1) + \frac{R e^{-2\eta}}{4} (e^{4U} \Gamma^2 + R^2 H_t^2). \quad (21)$$

The right hand side is positive since  $\rho \geq P_1$  and we can conclude, arguing as in Step 1 in Section 4 of [A1], that both  $R_t$  and  $|R_\theta|$  are bounded into the past. Hence  $R$  is uniformly  $C^1$  bounded to the past of the initial surface.

*Step 2.* (Bounds on  $U, A$  and  $\eta$  and their first derivatives.)

The bounds on  $U_t, A_t, U_\theta$  and  $A_\theta$  to the past of the initial surface are obtained by a light-cone estimate, which in this case, with one spatial dimension, is an application of the Gronwall method on two independent null paths. Then, by combining these results, one obtains the desired estimate.

The functions involved in the light-cone argument are quadratic functions in the first order derivatives of  $U$  and  $A$ , defined by

$$X = \frac{1}{2}R(U_t^2 + U_\theta^2) + \frac{e^{4U}}{8R}(A_t^2 + A_\theta^2), \quad (22)$$

$$Y = RU_tU_\theta + \frac{e^{4U}}{4R}A_tA_\theta. \quad (23)$$

We will see below that if we let the vector fields along the null paths act on  $X + Y$  and  $X - Y$  we obtain equations appropriate for applying a Gronwall argument.

Let us now derive bounds on  $U$  and  $A$  and their first order derivatives. By using the evolution equations (5) and (6) we find

$$\begin{aligned} \partial_\lambda(X + Y) &= \frac{-1}{2\sqrt{2}}R_\xi \left( U_t^2 - U_\theta^2 + \frac{e^{4U}}{4R^2}(-A_t^2 + A_\theta^2) \right) \\ &\quad + \frac{R}{2}U_\xi \left( e^{-2\eta+4U}\Gamma^2 + e^{2(\eta-U)}(\rho - P_1 + P_2 - P_3) \right) \\ &\quad + \frac{e^{2U}}{2R}A_\xi(R^2e^{2(U-\eta)}\Gamma H_t + 2Re^{2(\eta-U)}S_{23}), \end{aligned}$$

and

$$\begin{aligned} \partial_\xi(X - Y) &= \frac{-1}{2\sqrt{2}}R_\lambda \left( U_t^2 - U_\theta^2 + \frac{e^{4U}}{4R^2}(-A_t^2 + A_\theta^2) \right) \\ &\quad + \frac{R}{2}U_\lambda \left( e^{-2\eta+4U}\Gamma^2 + e^{2(\eta-U)}(\rho - P_1 + P_2 - P_3) \right) \\ &\quad + \frac{e^{2U}}{2R}A_\lambda(R^2e^{2(U-\eta)}\Gamma H_t + 2Re^{2(\eta-U)}S_{23}). \end{aligned}$$

It turns out that  $X$  and  $Y$  can be bounded by integrating these equations along null paths starting at a general point  $(t_1, \theta)$  in the past of the initial hypersurface and ending at the initial  $t_0$  surface  $t = t_0$ . The boundedness of the integrals which arise follows from the equations (20) and (21) and

the boundedness of  $R_\xi$  and  $R_\lambda$  in a similar way to the corresponding step in Section 4 of [A1]. Up to multiplication by functions which are already bounded and functions which can be bounded linearly in terms of  $X$  the terms involving the twist whose integrals must be estimated are  $e^{-2\eta+4U}\Gamma^2$  and  $e^{-2\eta+4U}\Gamma H_t$ . The first of these, multiplied by  $R/4$ , occurs as one of the positive summands on the right hand side of (20) and (21). Since by assumption  $R$  is bounded below the first term can be controlled. For the second we can use the elementary inequality

$$e^{2(U-\eta)}\Gamma H_t \leq \frac{1}{2}e^{-2\eta}(e^{4U}\Gamma^2 + H_t^2) \quad (24)$$

and the occurrence of  $\frac{R^3 e^{-2\eta}}{4}H_t^2$  as a summand in (20) and (21). Bounds on  $X$  and  $Y$  follow by applying Gronwall's inequality as in [A1]. Thus, as long as  $R$  stays uniformly bounded away from zero we conclude that  $U$  and its first order derivatives, and thus also  $A$  and its first order derivatives, are bounded. Bounds on  $|\eta|$ ,  $|\eta_t|$  and  $|\eta_\theta|$  are obtained in a similar way since the evolution equation (8) can be written

$$2\partial_\lambda\eta_\xi = U_\theta^2 - U_t^2 + \frac{e^{4U}}{4R^2}(A_t^2 - A_\theta^2) - \frac{e^{-2\eta+4U}}{4}\Gamma^2 - \frac{3R^2 e^{-2\eta}}{4}H_t^2 - e^{2(\eta-U)}P_3, \quad (25)$$

or equivalently,

$$2\partial_\xi\eta_\lambda = U_\theta^2 - U_t^2 + \frac{e^{4U}}{4R^2}(A_t^2 - A_\theta^2) - \frac{e^{-2\eta+4U}}{4}\Gamma^2 - \frac{3R^2 e^{-2\eta}}{4}H_t^2 - e^{2(\eta-U)}P_3. \quad (26)$$

The integrals of the right hand sides of these equations along null paths are bounded since  $P_3 \leq \rho - P_1$ . The terms involving  $\Gamma$  and  $H_t$  can be handled as above. Thus we find that  $\eta$  is uniformly  $C^1$  bounded to the past of the initial surface as long as  $R$  stays bounded away from zero.

*Step 3.* (Bound on the support of the momentum.)

Note that a solution  $f$  to the Vlasov equation is given by

$$f(t, \theta, v) = f_0(\Theta(0, t, \theta, v), V(0, t, \theta, v)), \quad (27)$$

where  $\Theta$  and  $V$  are solutions to the characteristic system

$$\begin{aligned} \frac{d\Theta}{ds} &= \frac{V^1}{V^0}, \\ \frac{dV^1}{ds} &= -(\eta_\theta - U_\theta)V^0 - (\eta_t - U_t)V^1 + U_\theta \frac{(V^2)^2}{V^0} \end{aligned}$$

$$\begin{aligned}
& -(U_\theta - \frac{R_\theta}{R}) \frac{(V^3)^2}{V^0} + \frac{A_\theta}{R} e^{2U} \frac{V^2 V^3}{V^0} \\
& - e^{-\eta} (e^{2U} \Gamma V^2 + R H_t V^3), \\
\frac{dV^2}{ds} &= -U_t V^2 - U_\theta \frac{V^1 V^2}{V^0}, \\
\frac{dV^3}{ds} &= -(\frac{R_t}{R} - U_t) V^3 + (U_\theta - \frac{R_\theta}{R}) \frac{V^1 V^3}{V^0} \\
& - \frac{e^{2U}}{R} (A_t + A_\theta \frac{V^1}{V^0}) V^2,
\end{aligned}$$

and  $\Theta(s, t, x, v)$ ,  $V(s, t, x, v)$  is the solution that goes through the point  $(\theta, v)$  at time  $t$ . Let us recall the definition of

$$Q(t) := \sup\{|v| : \exists (s, \theta) \in [t, t_0] \times S^1 \text{ such that } f(s, \theta, v) \neq 0\}.$$

We also define  $Q^j$ ,  $j = 1, 2, 3$  in the obvious way where  $|v|$  is replaced by  $|v^j|$ . If  $Q(t)$  can be controlled we obtain immediately from (14)-(16) bounds on  $\rho, J_k, P_k$ , and  $S_{jk}$ ,  $j, k = 1, 2, 3$ ,  $j \neq k$ , since  $\|f\|_\infty \leq \|f_0\|_\infty$  from (27). First we note that the quantities

$$e^U V^2, A e^U V^2 + R e^{-U} V^3$$

are conserved which a simple computation shows by using the equations for  $dV^2/ds$  and  $dV^3/ds$  above. More generally this is a consequence of the fact that if  $\gamma$  is a geodesic and  $k$  a Killing field then  $g(\gamma', k)$  is conserved along the geodesic. Here  $\gamma'$  is the tangent vector to  $\gamma$  and since the particles follow the geodesics of spacetime the tangent vector can be expressed in terms of  $v^\mu$ . We chose  $k = \partial_x$  and  $k = \partial_y$  as Killing fields to derive the conserved quantities given above. Since  $U$  and  $A$  are uniformly bounded as long as  $R$  stays bounded away from zero we conclude that  $V^2$  and  $V^3$  and thus  $Q^2$  and  $Q^3$  are bounded as well. In order to bound  $Q(t)$  we need to control the remaining component  $Q^1$ . From the auxiliary equations (10) and (12) we conclude that  $\Gamma$  and  $H_t$  can be bounded by the quantities involving  $\eta, U, A_t$ , which are known to be bounded, and by  $|S_{12}|$  and  $|S_{13}|$ . Since  $Q^2$  and  $Q^3$  are controlled we immediately get from (17) that  $|S_{12}|$  and  $|S_{13}|$  are bounded by a constant times  $Q^1$ . We conclude that as long as  $R$  stays bounded away from zero the terms involving  $\Gamma$  and  $H_t$  in the characteristic equation for  $dV^1/ds$  can be estimated by  $C(t)Q^1(t)$ , where  $C(t)$  is uniformly bounded on closed time intervals. Now, since the field components  $U, A$  and  $\eta$  and their first derivatives are known to be bounded on  $(t_-, t_0]$  (as long as  $R$  stays

bounded away from zero) we obtain from the characteristic equation for  $V^1$

$$|V^1(t)| \leq |V^1(t_0)| + C(t) \int_t^{t_0} Q^1(s) ds \leq |Q^1(t_0)| + C(t) \int_t^{t_0} Q^1(s) ds, \quad t < t_0.$$

Note that  $Q^1(t_0)$  is bounded by a positive constant since  $f_0$  has compact support. This inequality leads to a Gronwall inequality for  $Q^1(t)$  and we conclude that  $Q^1(t)$  is uniformly bounded on  $[t, t_0]$ .

Thus all the field components, their first derivatives and the matter terms are known to be bounded on  $(t_-, t_0]$ , as long as  $R$  stays bounded away from zero.

*Step 4.* (Bounds on the second order derivatives of the field components and on the first order derivatives of  $f$ .) From the Einstein-matter constraint equations in conformal coordinates we can express  $R_{t\theta}$  and  $R_{\theta\theta}$  in terms of uniformly bounded quantities, as long as  $R$  stays bounded away from zero. Therefore these functions are uniformly bounded and equation (7) then implies that  $R_{tt}$  is uniformly bounded as well.

In the vacuum case one can take the derivative of the evolution equations and repeat the argument in Step 2 to obtain bounds on second order derivatives of  $U$  and  $A$ . Here we need another argument. First we write the evolution equations for  $U$  and  $A$  in the forms

$$\begin{aligned} U_{tt} - U_{\theta\theta} &= \frac{(R_\theta - R_t)}{2R}(U_\theta + U_t) - \frac{(R_\theta + R_t)}{2R}(U_t - U_\theta) \\ &\quad + \frac{e^{4U}}{2R^2}(A_t - A_\theta)(A_t + A_\theta) + \frac{e^{-2\eta+4U}}{2}\Gamma^2 + \frac{1}{2}e^{2(\eta-U)}\kappa, \end{aligned}$$

and

$$\begin{aligned} A_{tt} - A_{\theta\theta} &= \frac{(R_t - R_\theta)}{2R}(A_\theta + A_t) + \frac{(R_\theta + R_t)}{2R}(A_t - A_\theta) \\ &\quad - 2(A_t - A_\theta)(U_\theta + U_t) - 2(A_\theta + A_t)(U_t - U_\theta) \\ &\quad + R^2 e^{-2\eta} \Gamma H_t + 2R e^{2(\eta-2U)} S_{23}, \end{aligned}$$

where  $\kappa$  denotes  $\rho - P_1 + P_2 - P_3$ . Taking the  $\theta$ -derivative of these equations gives

$$\begin{aligned} \partial_\lambda \partial_\xi U_\theta &= L + \frac{R_\lambda}{2R} \partial_\xi U_\theta + \frac{R_\xi}{2R} \partial_\lambda U_\theta + \frac{e^{4U}}{2R^2} (A_\lambda \partial_\xi A_\theta + A_\xi \partial_\lambda A_\theta) \\ &\quad + \frac{1}{2} (e^{-2\eta+4U} \Gamma_\theta + e^{2(\eta-U)} \kappa_\theta), \end{aligned} \tag{28}$$

and

$$\begin{aligned} \partial_\lambda \partial_\xi A_\theta &= L + \frac{R_\lambda}{2R} \partial_\xi A_\theta - \frac{R_\xi}{2R} \partial_\lambda A_\theta + 2U_\xi \partial_\lambda A_\theta + 2A_\lambda \partial_\xi U_\theta + 2U_\lambda \partial_\xi A_\theta \\ &\quad + 2A_\xi \partial_\lambda U_\theta + R^2 e^{-2\eta} (\Gamma H_t)_\theta + 2R e^{2(\eta-2U)} (S_{23})_\theta, \end{aligned} \quad (29)$$

Here,  $L$  contains only  $\Gamma, H_t, \kappa$  and  $S_{23}$ , first order derivatives of  $U, A$  and  $\eta$ , and first and second order derivatives of  $R$ , which all are known to be bounded. These equations can of course also be written in a form where the left hand sides read  $\partial_\xi \partial_\lambda U_\theta$  and  $\partial_\xi \partial_\lambda A_\theta$ , respectively. By integrating these equations along null paths to the past of the initial surface, we get from a Gronwall argument a bound on

$$\sup_{\theta \in S^1} (|\partial_\xi U_\theta| + |\partial_\lambda U_\theta| + |\partial_\xi A_\theta| + |\partial_\lambda A_\theta|),$$

as long as  $R$  is bounded away from zero, under the hypothesis that the integral of the differentiated terms  $\Gamma_\theta, H_{t\theta}, \kappa_\theta$  and  $(S_{23})_\theta$  can be controlled. That the first two terms are bounded follows immediately from the auxiliary equations (9) and (11) in view of the bound on  $Q(t)$ . In order to bound the matter terms we make use of a device introduced by Glassey and Strauss [GS] for treating the Vlasov-Maxwell equation. Since there is no essential difference to the case treated in [A1] we do not repeat the proof here. This procedure allows the integrals of the differentiated matter terms to be controlled and the Gronwall argument referred to above goes through. So we obtain uniform bounds on  $|\partial_\xi U_\theta|, |\partial_\lambda U_\theta|, |\partial_\xi A_\theta|$ , and  $|\partial_\lambda A_\theta|$ , and therefore also on  $|U_{\theta\theta}|, |U_{t\theta}|, |A_{\theta\theta}|$  and  $|A_{t\theta}|$ , as long as  $R$  is bounded away from zero. The evolution equations (5) and (6) then give uniform bounds on  $|U_{tt}|$  and  $|A_{tt}|$ . Bounds on second order derivatives of  $G$  and  $H$  follow from the auxiliary equations. By differentiating equation (8), it is now straightforward to obtain bounds on the second order derivatives of  $\eta$ , using similar arguments to those already discussed here, in particular the integrals involving matter quantities can be treated as above. Bounds on the first order derivatives of the distribution function  $f$  may now be obtained from the known bounds on the field components from the formula

$$f(t, \theta, v) = f_0(\Theta(0, t, \theta, v), V(0, t, \theta, v)), \quad (30)$$

since  $f_0$  is smooth and since  $\partial\Theta$  and  $\partial V$  (here  $\partial$  denotes  $\partial_t, \partial_\theta$  or  $\partial_v$ ) can be controlled by a Gronwall argument in view of the characteristic system and the auxiliary equations (9)-(12).

*Step 5.* (Bounds on higher order derivatives and completion of the proof.)

It is clear that the method described above can be continued for obtaining bounds on higher derivatives as well. Hence, we have uniform bounds on the functions  $R, U, A, \eta$  and  $f$  and all their derivatives on the interval  $(t_-, t_0]$  if  $R > \epsilon > 0$ . This implies that the solution extends to  $t \rightarrow -\infty$  as long as  $R$  stays bounded away from zero.

Later we will require a slight generalization of these results in order to show that the arguments of Section 5 of [ARR] generalize to cover the case of  $T^2$  symmetry. Once it has been established that  $\nabla R$  is timelike, the estimates in the later steps hold for any subset  $Z$  of the half-plane  $t \leq t_0$  provided  $Z$  is a future set. By definition this means that any future directed causal curve in the region  $t \leq t_0$  starting at a point of  $Z$  remains in  $Z$ . (For information on concepts such as this concerning causal structures see e.g. [HE].) Thus if  $R$  is bounded away from zero on  $Z$  and  $t$  is bounded on  $Z$  then all the unknowns and their derivatives can be controlled on  $Z$ .

Now consider a special choice of the subset  $Z$ , namely that which is defined by the inequalities  $t_1 < t \leq t_0$  and  $\theta_1 + t_0 - t < \theta < \theta_2 - t_0 + t$  for some numbers  $\theta_1, \theta_2$  and  $t_1$  satisfying the inequalities  $\theta_1 < \theta_2$  and  $t_1 > t_0 - (1/2)(\theta_2 - \theta_1)$ . Suppose a solution of the Einstein-Vlasov system in conformal coordinates defined on this region is such that  $R$  is bounded away from zero. Then the functions defining the solution extend smoothly to the boundary of  $Z$  at  $t = t_1$ . They define smooth Cauchy data for the Einstein-Vlasov system. Applying the standard local existence theorem (without symmetry) allows the solution to be extended through that boundary. Repeating the construction of conformal coordinates in Section 2 then shows that we get an extension of the solution written in conformal coordinates through that boundary.

## 4 Analysis in the expanding direction

In this section we want to investigate the Einstein-Vlasov system with  $T^2$  symmetry in the expanding direction. We write the system in areal coordinates, i.e., the coordinates are chosen such that  $R = t$ . The circumstances under which coordinates of this type exist are discussed in Section 5. We prove that for initial data on a hypersurface of constant time corresponding solutions exist for all future time with respect to the areal time coordinate. In order to extend a solution defined on a finite time interval in these coordinates to one which exists globally in time it is sufficient to obtain uniform bounds on the field components and the distribution function and all their

derivatives on a finite time interval  $[t_0, t_+)$  on which the local solution exists. For in this case the functions defining the solution are uniformly continuous and extend continuously to  $t = t_+$ . Since all derivatives are bounded the extension is smooth and we obtain new initial data on  $t = t_+$ . Applying the general local existence theorem for the Einstein-Vlasov system [CB] allows the spacetime to be extended beyond  $t = t_+$ . Conformal coordinates can be introduced on a neighbourhood of  $t = t_+$  and from those areal coordinates can be constructed. In this way an extension to the future of the solution written in areal coordinates is obtained.

Below the form of the metric and the Einstein-Vlasov system are given in areal coordinates. The functions  $\alpha, \eta, U, A, G, H$  all depend on  $t$  and  $\theta$  and the function  $f$  depends on  $t, \theta$  and  $v \in \mathbb{R}^3$  and as before we have set  $\Gamma = G_t + AH_t$ . The orthonormal frame used to parametrize  $PM$  is

$$\alpha^{1/2} e^{U-\eta} \frac{\partial}{\partial t}, e^{U-\eta} \left( \frac{\partial}{\partial \theta} - G \frac{\partial}{\partial x} - H \frac{\partial}{\partial y} \right), e^{-U} \frac{\partial}{\partial x}, e^U t^{-1} \left( \frac{\partial}{\partial y} - A \frac{\partial}{\partial x} \right)$$

Metric

$$g = e^{2(\eta-U)} (-\alpha dt^2 + d\theta^2) + e^{2U} [dx + A dy + (G + AH) d\theta]^2 + e^{-2U} t^2 [dy + H d\theta]^2 \quad (31)$$

The Einstein-matter constraint equations

$$\begin{aligned} \frac{\eta_t}{t} &= U_t^2 + \alpha U_\theta^2 + \frac{e^{4U}}{4t^2} (A_t^2 + \alpha A_\theta^2) + \frac{e^{-2\eta}}{4} (e^{4U} \Gamma^2 + t^2 H_t^2) \\ &\quad + e^{2(\eta-U)} \alpha \rho \end{aligned} \quad (32)$$

$$\frac{\eta_\theta}{t} = 2U_t U_\theta + \frac{e^{4U}}{2t^2} A_t A_\theta - \frac{\alpha_\theta}{2t\alpha} - e^{2(\eta-U)} \sqrt{\alpha} J_1 \quad (33)$$

$$\alpha_t = 2t\alpha^2 e^{2(\eta-U)} (P_1 - \rho) - \alpha t e^{-2\eta} (e^{4U} \Gamma^2 + t^2 H_t^2) \quad (34)$$

The Einstein-matter evolution equations

$$\begin{aligned} \eta_{tt} - \alpha \eta_{\theta\theta} &= \frac{\eta_\theta \alpha_\theta}{2} + \frac{\eta_t \alpha_t}{2\alpha} - \frac{\alpha_\theta^2}{4\alpha} + \frac{\alpha_{\theta\theta}}{2} - U_t^2 + \alpha U_\theta^2 \\ &\quad + \frac{e^{4U}}{4t^2} (A_t^2 - \alpha A_\theta^2) - \frac{3}{4} e^{-2\eta} t^2 H_t^2 - \frac{1}{4} e^{-2\eta} e^{4U} \Gamma^2 \end{aligned}$$



$$-\alpha e^{2(\eta-U)} P_3, \quad (35)$$

$$\begin{aligned} U_{tt} - \alpha U_{\theta\theta} &= -\frac{U_t}{t} + \frac{U_\theta \alpha_\theta}{2} + \frac{U_t \alpha_t}{2\alpha} + \frac{e^{4U}}{2t^2} (A_t^2 - \alpha A_\theta^2) \\ &\quad + \frac{e^{-2\eta} e^{4U}}{2} \Gamma^2 + \frac{e^{2(\eta-U)} \alpha}{2} (\rho - P_1 + P_2 - P_3) \end{aligned} \quad (36)$$

$$\begin{aligned} A_{tt} - \alpha A_{\theta\theta} &= \frac{A_t}{t} + \frac{\alpha_\theta A_\theta}{2} + \frac{\alpha_t A_t}{2\alpha} - 4A_t U_t + 4\alpha A_\theta U_\theta \\ &\quad + t^2 e^{-2\eta} \Gamma H_t + 2t\alpha e^{2(\eta-2U)} S_{23}. \end{aligned} \quad (37)$$

Auxiliary equations

$$\partial_\theta [e^{-2\eta} \alpha^{-1/2} e^{4U} \Gamma] = -2e^\eta J_2, \quad (38)$$

$$\partial_t [e^{-2\eta} t \alpha^{-1/2} e^{4U} \Gamma] = 2t \alpha^{1/2} e^\eta S_{12}, \quad (39)$$

$$\partial_\theta [e^{-2\eta} \alpha^{-1/2} (A e^{4U} \Gamma + t^2 H_t)] = -2e^\eta A J_2 - 2t e^{\eta-2U} J_3, \quad (40)$$

$$\partial_t [e^{-2\eta} t \alpha^{-1/2} (A e^{4U} \Gamma + t^2 H_t)] = 2t \alpha^{1/2} e^\eta (A S_{12} + t e^{-2U} S_{13}). \quad (41)$$

The Vlasov equation

$$\begin{aligned} &\frac{\partial f}{\partial t} + \frac{\sqrt{\alpha} v^1}{v^0} \frac{\partial f}{\partial \theta} - \left[ (\eta_\theta - U_\theta + \frac{\alpha_\theta}{2\alpha}) \sqrt{\alpha} v^0 + (\eta_t - U_t) v^1 \right. \\ &\quad \left. - \frac{\sqrt{\alpha} e^{2U} A_\theta}{t} \frac{v^2 v^3}{v^0} + \frac{\sqrt{\alpha} U_\theta}{v^0} ((v^3)^2 - (v^2)^2) + e^{-\eta} (e^{2U} \Gamma v^2 + t H_t v^3) \right] \frac{\partial f}{\partial v^1} \\ &\quad - \left[ U_t v^2 + \sqrt{\alpha} U_\theta \frac{v^1 v^2}{v^0} \right] \frac{\partial f}{\partial v^2} \\ &\quad - \left[ \left( \frac{1}{t} - U_t \right) v^3 - \sqrt{\alpha} U_\theta \frac{v^1 v^3}{v^0} + \frac{e^{2U} v^2}{t} (A_t + \sqrt{\alpha} A_\theta \frac{v^1}{v^0}) \right] \frac{\partial f}{\partial v^3} = 0. \end{aligned} \quad (42)$$

The matter quantities are defined as in (14)-(17).

*Step 1.* (Bounds on  $\alpha, U, A, G, H$  and  $\tilde{\eta}$ .)

In this step we first show an “energy” monotonicity lemma and then we show how this result leads to bounds on  $\tilde{\eta} := \eta + \ln \alpha/2$  and on  $U$  and  $A$ . Let  $E(t)$  be defined by

$$\begin{aligned} E(t) &= \int_{S^1} \left[ \alpha^{-\frac{1}{2}} U_t^2 + \sqrt{\alpha} U_\theta^2 + \frac{e^{4U}}{4t^2} (\alpha^{-\frac{1}{2}} A_t^2 + \sqrt{\alpha} A_\theta^2) \right. \\ &\quad \left. + \frac{e^{-2\eta} \alpha^{-1/2}}{4} (e^{4U} \Gamma^2 + t^2 H_t^2) + \sqrt{\alpha} e^{2(\eta-U)} \rho \right] d\theta. \end{aligned} \quad (43)$$

**Lemma 1**  $E(t)$  is a monotonically decreasing function in  $t$ , and satisfies

$$\begin{aligned} \frac{d}{dt}E(t) = & -\frac{2}{t} \int_{S^1} \left[ \frac{U_t^2}{\sqrt{\alpha}} + \frac{e^{4U}}{4t^2} \sqrt{\alpha} A_\theta^2 + \frac{e^{-2\eta}}{4\sqrt{\alpha}} (e^{4U} \Gamma^2 + 2t^2 H_t^2) \right. \\ & \left. + \frac{\sqrt{\alpha}}{2} e^{2(\eta-U)} (\rho + P_3) \right] d\theta \leq 0. \end{aligned} \quad (44)$$

*Proof.* This is a straightforward but lengthy computation. Let us sketch the steps involved. After taking the time derivative of the integrand we use the evolution equations for  $U$  and  $A$  to substitute for the second order derivatives, we use the auxiliary equations to express second order derivatives of  $G$  and  $H$  in terms of matter quantities and we express  $\rho_t$  by using the Vlasov equation. Integrating by parts and using the constraint equations for  $\eta_t$  and  $\alpha_t$  lead to (44).

□

Let us now define the quantity  $\tilde{\eta}$  by

$$\tilde{\eta} = \eta + \frac{1}{2} \ln \alpha. \quad (45)$$

The difference  $\tilde{\eta}(t, \theta_1) - \tilde{\eta}(t, \theta_2)$  can be estimated by integrating the expression for  $\tilde{\eta}_\theta$  resulting from (33) and using the energy bound. Since (33) does not contain the twist quantities this is exactly as in [A1]. The next step is to use the constraint equations (32) and (34) to bound the integral  $\int_{S^1} \tilde{\eta} d\theta$ . The net contribution to  $\tilde{\eta}_t$  from the twist quantities has the opposite sign from that of the other terms. So when obtaining an upper bound for the integral it can be discarded and the argument proceeds as in [A1]. On the other hand the twist terms must be taken into account when obtaining a lower bound for the integral. The expression which has to be estimated is equal to the twist contribution to the energy up to a factor  $\alpha$ . This factor is bounded since  $\alpha$  is monotone decreasing. Knowing that the difference of  $\tilde{\eta}$  at two points  $\theta_1$  and  $\theta_2$  at any given time and its integral at any given time are bounded it follows that  $\tilde{\eta}(t, \theta) \leq C(t)$  for some bounded function  $C(t)$ .

**Remark.** In the analysis below  $C(t)$  will always denote a uniformly bounded function on  $[t_0, t_+)$ . Sometimes we introduce other functions with the same property only for the purpose of trying to make some estimates become more transparent.

Next we show that the boundedness of  $E(t)$ , together with the constraint equation (34), lead to a bound on  $|U|$ . For any  $\theta_1, \theta_2 \in S^1$ , and  $t \in [t_0, t_+)$  we get by Hölder's inequality

$$\begin{aligned} |U(t, \theta_2) - U(t, \theta_1)| &= \left| \int_{\theta_1}^{\theta_2} U_\theta(t, \theta) d\theta \right| \\ &\leq \left( \int_{\theta_1}^{\theta_2} \alpha^{-1/2} d\theta \right)^{1/2} \left( \int_{\theta_1}^{\theta_2} \sqrt{\alpha} U_\theta^2 d\theta \right)^{1/2}. \end{aligned} \quad (46)$$

The second factor on the right hand side is clearly bounded by  $(E(t_0))^{1/2}$ . For the first factor we use the constraint equation (34). This equation can be written as

$$\partial_t(\alpha^{-1/2}) = t\sqrt{\alpha}e^{2(\eta-U)}(\rho - P_1) + t\alpha^{-1/2}\frac{e^{-2\eta}}{2}(e^{4U}\Gamma^2 + t^2H_t^2), \quad (47)$$

so that for  $t \in [t_0, t_+)$

$$\begin{aligned} \alpha^{-1/2}(t, \theta) &= \int_{t_0}^t s\sqrt{\alpha}e^{2(\eta-U)}(\rho - P_1) + s\alpha^{-1/2}\frac{e^{-2\eta}}{2}(e^{4U}\Gamma^2 + s^2H_t^2)ds \\ &\quad + \alpha^{-1/2}(t_0, \theta). \end{aligned} \quad (48)$$

Since  $\rho \geq P_1$ , the integrand is positive and bounded by a multiple of the integrand of  $E(t)$ . Letting  $C$  denote the supremum of  $\alpha^{-1/2}(t_0, \cdot)$  over  $S^1$  we get

$$\int_{\theta_1}^{\theta_2} \alpha^{-1/2} d\theta \leq \int_{t_0}^t 2sE(s)ds + 2\pi C \leq 2E(t_0)(t^2 - t_0^2)/2 + 2\pi C.$$

Hence, for any  $\theta_1, \theta_2 \in S^1$  we have

$$|U(t, \theta_2) - U(t, \theta_1)| \leq C(t). \quad (49)$$

Next  $\int_{S^1} U(t, \theta) d\theta$  can be estimated using the energy just as in [A1] since the twist quantities do not play a role in that argument. Knowing that the difference of  $U$  at any two spatial points and the modulus of its integral over the circle are bounded we can conclude that  $U$  itself is bounded. These arguments also apply to  $A$ , since the factor  $e^{4U}$  is controlled by the uniform bound on  $U$ . Bounds on  $e^{-\eta}\Gamma$  and  $e^{-\eta}H_t$  also follow from these arguments. Indeed, let  $P = e^{4U-2\eta}\alpha^{-1/2}\Gamma$ . We have from the auxiliary equation (38) an

expression for  $P_\theta$  in terms of  $J_2$  and we get

$$\begin{aligned} |P(t, \theta_2) - P(t, \theta_1)| &= \left| \int_{\theta_1}^{\theta_2} P_\theta(t, \theta) d\theta \right| \leq \int_{\theta_1}^{\theta_2} 2e^\eta |J_2| d\theta \\ &\leq 2\|e^{-\tilde{\eta}+2U}\|_\infty \int_{\theta_1}^{\theta_2} \sqrt{\alpha} e^{2(\eta-U)} |J_2| d\theta \leq C(t)E(t). \end{aligned} \quad (50)$$

Note that  $\tilde{\eta}$  and  $U$  are known to be bounded. Similarly, using the auxiliary equation (39) we obtain a bound on

$$\left| \int_{S^1} P(t, \theta) d\theta \right|$$

in the same spirit as for  $U$ . This leads to a bound on  $P$  itself which implies that  $e^{-\eta}\Gamma$  is bounded. A bound on  $e^{-\eta}H_t$  follows if we instead let  $P = e^{-2\eta}\alpha^{-1/2}t^2H_t$ . Using the auxiliary equations (38-41) we get expressions on  $P_\theta$  and  $P_t$ . In the expression for  $P_t$  we get a term containing  $A_t$  which we treat as in the case of bounding  $U$ , i.e. we use a Hölder argument to bound that term in terms of  $E(t)$ . We also use that  $e^{-\eta}\Gamma$ ,  $A$  and  $\tilde{\eta}$  are bounded but all ideas have already been used above so we leave them out and conclude that  $e^{-\eta}|H_t|$  is bounded on  $[t_0, t_+)$  as well.

*Step 2.* (Bounds on  $U_t, U_\theta, A_t, A_\theta, \eta_t, \tilde{\eta}_\theta, \alpha_t$  and  $Q(t)$ .)

To bound the derivatives of  $U$  we use light-cone estimates in a similar way as for the contracting direction. However, the matter terms must be treated differently and we need to carry out a careful analysis of the characteristic system associated with the Vlasov equation. Let us define

$$X = \frac{1}{2}(U_t^2 + \alpha U_\theta^2) + \frac{e^{4U}}{8t^2}(A_t^2 + \alpha A_\theta^2), \quad (51)$$

$$Y = \sqrt{\alpha}U_tU_\theta + \frac{e^{4U}}{4t^2}A_tA_\theta, \quad (52)$$

and

$$\chi = \frac{1}{\sqrt{2}}(\partial_t + \sqrt{\alpha}\partial_\theta) \quad (53)$$

$$\zeta = \frac{1}{\sqrt{2}}(\partial_t - \sqrt{\alpha}\partial_\theta) \quad (54)$$

A motivation for the introduction of these quantities is based on similar arguments as those given in Step 1, Section 4. For details we refer to [BCIM].

**Remark.** We use the same notations,  $X$  and  $Y$ , as in the contracting direction, and below we continue to carry over the notations. The analysis

in the respective direction is independent so there should be no risk of confusion.

By using the evolution equation (36), a short computation shows that

$$\begin{aligned}
\zeta(X + Y) &= \frac{\alpha_t}{2\sqrt{2}\alpha}(X + Y) \\
&\quad - \frac{1}{\sqrt{2}t} \left( U_t^2 + \sqrt{\alpha}U_tU_\theta + \frac{e^{4U}}{4t^2}(\alpha A_\theta^2 + \sqrt{\alpha}A_tA_\theta) \right) \\
&\quad + \frac{(U_t + \sqrt{\alpha}U_\theta)}{2\sqrt{2}}(e^{4U-2\eta}\Gamma^2 + \alpha e^{2(\eta-U)}\kappa) \\
&\quad + \frac{(A_t + \sqrt{\alpha}A_\theta)}{4\sqrt{2}}(e^{4U-2\eta}\Gamma H_t + 2\alpha e^{2\eta}S_{23}),
\end{aligned} \tag{55}$$

$$\begin{aligned}
\chi(X - Y) &= \frac{\alpha_t}{2\sqrt{2}\alpha}(X - Y) \\
&\quad - \frac{1}{\sqrt{2}t} \left( U_t^2 - \sqrt{\alpha}U_tU_\theta + \frac{e^{4U}}{4t^2}(\alpha A_\theta^2 - \sqrt{\alpha}A_tA_\theta) \right) \\
&\quad + \frac{(U_t - \sqrt{\alpha}U_\theta)}{2\sqrt{2}}(e^{4U-2\eta}\Gamma^2 + \alpha e^{2(\eta-U)}\kappa) \\
&\quad + \frac{(A_t - \sqrt{\alpha}A_\theta)}{4\sqrt{2}}(e^{4U-2\eta}\Gamma H_t + 2\alpha e^{2\eta}S_{23}),
\end{aligned} \tag{56}$$

Here  $\kappa = \rho - P_1 + P_2 - P_3$ . Now we wish to integrate these equations along the integral curves of the vector fields  $\chi$  and  $\zeta$  respectively (let us henceforth call these integral curves null curves, since they are null with respect to the two-dimensional “base spacetime”). Below we show that the quantity

$$W(t) := \sup_{\theta \in S^1} X(t, \cdot) + Q^2(t), \tag{57}$$

is uniformly bounded on  $[t_0, t_+)$  by deriving the inequality

$$W(t) \leq C + \int_{t_0}^t W(s) \ln W(s) ds. \tag{58}$$

First we note, exactly as in the contracting direction, that the symmetry implies that

$$V^2(t)e^{U(t, \Theta(t))}$$

and

$$V^2(t)Ae^U + V^3(t)te^{-U(t,\Theta(t))},$$

are conserved. Here  $V^2(t)$ ,  $V^3(t)$  and  $\Theta(t)$  are solutions to the characteristic system associated to the Vlasov equation. From Step 2 we have that  $U$  and  $A$  are uniformly bounded on  $[t_0, t_+)$ . Hence  $|V^2(t)|$  and  $|V^3(t)|$  are both uniformly bounded on  $[t_0, t_+)$ , and since the initial distribution function  $f_0$  has compact support we conclude that

$$\sup\{|v^2| + |v^3| : \exists(s, \theta) \in [t_0, t] \times S^1 \text{ with } f(s, \theta, v) \neq 0\}, \quad (59)$$

is uniformly bounded on  $[t_0, t_+)$ . Therefore, in order to control  $Q(t)$  it is sufficient to control

$$Q^1(t) := \sup\{|v^1| : \exists(s, \theta) \in [t_0, t] \times S^1 \text{ such that } f(s, \theta, v) \neq 0\}. \quad (60)$$

Below we introduce the uniformly bounded function  $\gamma(t)$  to denote estimates regarding the variables  $v^2$  and  $v^3$ . As observed in [A1] there is some cancellation to take advantage of in the matter term  $(\rho - P_1)$  which appears in the equations for  $X + Y$  and  $X - Y$  above. It is proved there that

$$(\rho - P_1)(t, \theta) \leq C\gamma(t) \ln Q^1(t) \quad (61)$$

In a similar fashion we can estimate  $P_2, P_3$  and  $S_{23}$ .

Let us now derive (58). As in Step 2 in Section 4 we integrate the equations above for  $X + Y$  and  $X - Y$  along null paths. For  $t \geq t_0$  integrate along the two null paths defined by  $\chi$  and  $\zeta$ , starting at  $(t_0, \theta)$  and add the results. In this way the following inequality can be derived:

$$\sup_{\theta} X(t, \cdot) \leq C + C(t) \int_{t_0}^t [1 + \sup_{\theta} X(s, \cdot)] \ln Q^1(s) ds \quad (62)$$

In doing this it is important to use the fact that  $\alpha e^{2\eta} = e^{2\tilde{\eta}}$  and the above estimates for matter quantities in terms of  $\ln Q^1$ . What is new compared to [A1] is the occurrence of twist quantities and these can be treated using the fact that  $e^{-\eta}\Gamma$  and  $e^{-\eta}H_t$  are bounded.

Let us now derive an estimate for  $Q^1$  in terms of  $\sup_{\theta} X$ .

**Lemma 2** *Let  $Q^1(t)$  and  $X(t, \theta)$  be as above. Then*

$$|Q^1(t)|^2 \leq C + D(t) \int_{t_0}^t [(Q^1(s))^2 + \sup_{\theta} X(s, \cdot)] ds, \quad (63)$$

where  $C$  is a constant and  $D(t)$  is a uniformly bounded function on  $[t_0, t_+)$ .

*Proof.* The characteristic equation for  $V^1$  associated to the Vlasov equation and the constraint equations (32) and (33) imply that

$$\frac{d}{ds}(V^1(s))^2 = 2V^1(s)\frac{d}{ds}V^1(s) = T_1 + T_2 + T_3 + T_4, \quad (64)$$

Here  $T_1$ ,  $T_2$  and  $T_3$  are expressions given in [A1] and they can be estimated just as in the corresponding lemma of that reference. The additional term  $T_4$ , which collects together the contributions involving the twist quantities, is

$$T_4 = -2s(V^1)^2 \frac{e^{-2\eta}}{4} (e^{4U}\Gamma^2 + s^2 H_t^2) - 2V^1(s)[e^{-\eta}(e^{2U}\Gamma V^2 + sH_t V^3)] \quad (65)$$

It is easily seen that  $|T_4| \leq C\gamma(t)(Q^1(t))^2$  and this suffices to obtain the desired estimate.

□

Combining the estimate for  $(Q^1(t))^2$  in the lemma and the estimate (62) for  $\sup_\theta X(t, \cdot)$ , we find that  $W(t)$  satisfies the estimate (58) and is thus uniformly bounded. The constraint equation (32) now immediately shows that  $|\eta_t|$  is bounded by

$$2tX + \frac{te^{-2\eta}}{4}(e^{4U}\Gamma^2 + t^2 H_t^2) + te^{2(\tilde{\eta}-U)}\rho \leq C(t)[1 + \sup_\theta X(t, \cdot) + (Q(t))^3],$$

since

$$\rho = \int_{\mathbb{R}^3} f dv \leq \|f_0\|_\infty \int_{|v| \leq Q(t)} dv \leq C(Q(t))^3.$$

Recall that  $e^{-2\eta}(e^{4U}\Gamma^2 + t^2 H_t^2)$  is known to be bounded. Thus also  $\eta$  is bounded which implies that both  $|\Gamma|$  and  $|H_t|$  are bounded. The bound on  $\eta$  also provides a bound on  $|\tilde{\eta}_\theta|$  using the constraint equation (33), where  $|J_1|$  is estimated in terms of  $Q(t)$ . Analogous arguments show that  $|\alpha_t|$  is uniformly bounded. The uniform bound on  $X$  provides bounds on  $|U_t|$  and  $|A_t|$ , but to conclude that  $|U_\theta|$  and  $|A_\theta|$  are bounded we have to show that  $\alpha$  stays uniformly bounded away from zero. Equation (34) is easily solved,

$$\alpha(t, \theta) = \alpha(t_0, \theta)e^{\int_{t_0}^t F(s, \theta) ds}, \quad (66)$$

where

$$F(t, \theta) := -2te^{2(\tilde{\eta}-U)}(\rho - P_1) - te^{-2\eta}(e^{4U}\Gamma^2 + t^2 H_t^2),$$

which is uniformly bounded from below. Hence  $|U_\theta|$  and  $|A_\theta|$  are bounded and Step 2 is complete.

*Step 3.* (Bounds on  $\partial f$ ,  $\alpha_\theta$  and  $\eta_\theta$ .)

The main goal in this step is to show that the first derivatives of the distribution function are bounded. In view of the bound on  $Q(t)$  we then also obtain bounds on the first derivatives of the matter terms  $\rho$ ,  $J_k$ ,  $S_{jk}$  and  $P_k$ ,  $j, k = 1, 2, 3; j \neq k$ . Such bounds together with bounds on the  $\theta$  derivatives of the twist quantities almost immediately lead to bounds on  $\alpha_\theta$  and  $\eta_\theta$ .

Recall that the solution  $f$  can be written in the form

$$f(t, \theta, v) = f_0(\Theta(0, t, \theta, v), V(0, t, \theta, v)), \quad (67)$$

where  $\Theta(s, t, \theta, v), V(s, t, \theta, v)$  is the solution to the characteristic system

$$\frac{d\Theta}{ds} = \sqrt{\alpha} \frac{V^1}{V^0}, \quad (68)$$

$$\begin{aligned} \frac{dV^1(s)}{ds} = & -(\eta_\theta - U_\theta + \frac{\alpha_\theta}{2\alpha})\sqrt{\alpha}V^0 - (\eta_t - U_t)V^1 \\ & - \frac{\sqrt{\alpha}U_\theta}{V^0}((V^2)^2 - (V^3)^2) + \frac{\sqrt{\alpha}A_\theta}{sV^0}e^{2U}V^2V^3 \\ & - e^{-\eta}(e^{2U}\Gamma V^2 + sH_tV^3), \end{aligned} \quad (69)$$

$$\frac{dV^2}{ds} = -U_tV^2 - \sqrt{\alpha}U_\theta \frac{V^1V^2}{V^0}, \quad (70)$$

$$\begin{aligned} \frac{dV^3}{ds} = & -(\frac{1}{s} - U_t)V^3 + \sqrt{\alpha}U_\theta \frac{V^1V^3}{V^0} \\ & - \frac{e^{2U}}{s}(A_t + \sqrt{\alpha}A_\theta \frac{V^1}{V^0})V^2, \end{aligned} \quad (71)$$

with the property  $\Theta(t, t, \theta, v) = \theta$ ,  $V(t, t, \theta, v) = v$ . Hence, in order to establish bounds on the first derivatives of  $f$  it is sufficient to bound  $\partial\Theta$  and  $\partial V$  since  $f_0$  is smooth. Here  $\partial$  denotes the first order derivative with respect to  $t, \theta$  or  $v$ . Evolution equations for  $\partial\Theta$  and  $\partial V$  are provided by the characteristic system above. However, the right hand sides will contain second order derivatives of the field components, but so far we have only obtained bounds on the first order derivatives (except for  $\eta_\theta, \alpha_\theta$ ). Yet, certain combinations of second order derivatives can be controlled. Behind this observation lies a geometrical idea which plays a fundamental role in general relativity. An important property of curvature is its control over the relative behaviour of nearby geodesics. Let  $\gamma(u, \lambda)$  be a two-parameter family of geodesics, i.e. for each fixed  $\lambda$ , the curve  $u \mapsto \gamma(u, \lambda)$  is a geodesic. Define the variation



vector field  $Y := \gamma_\lambda(u, 0)$ . This vector field satisfies the geodesic deviation equation (or Jacobi equation) (see eg. [HE])

$$\frac{D^2 Y}{Du^2} = R_{Y\gamma'} \gamma', \quad (72)$$

where  $D/Du$  is the covariant derivative,  $R$  the Riemann curvature tensor, and  $\gamma' := \gamma_u(u, 0)$ . Now, the Einstein tensor is closely related to the curvature tensor and since the Einstein tensor is proportional to the energy momentum tensor which we can control from Step 2, it is meaningful, in view of (72) (with  $Y = \partial\Theta$ ), to look for linear combinations of  $\partial\Theta$  and  $\partial V$  which satisfy an equation with bounded coefficients. More precisely, we want to substitute the twice differentiated field components which appear by taking the derivative of the characteristic system by using the Einstein equations. The geodesic deviation equation has previously played an important role in studies of the Einstein-Vlasov system ([R14], [Rn] and [R13]).

**Lemma 3** *Let  $\Theta(s) = \Theta(s, t, \theta, v)$  and  $V^k(s) = V^k(s, t, \theta, v)$ ,  $k = 1, 2, 3$  be a solution to the characteristic system (68)-(71). Let  $\partial$  denote  $\partial_t, \partial_\theta$  or  $\partial_v$ , and define*

$$\Psi = \alpha^{-1/2} \partial\Theta, \quad (73)$$

$$\begin{aligned} Z^1 = \partial V^1 &+ \left( \frac{\eta_t V^0}{\sqrt{\alpha}} - \frac{U_t V^0}{\sqrt{\alpha}} \frac{(V^0)^2 - (V^1)^2 + (V^2)^2 - (V^3)^2}{(V^0)^2 - (V^1)^2} \right. \\ &+ U_\theta \frac{V^1((V^2)^2 - (V^3)^2)}{(V^0)^2 - (V^1)^2} - \frac{A_t e^{2U}}{\sqrt{\alpha} t} \frac{V^0 V^2 V^3}{(V^0)^2 - (V^1)^2} \\ &\left. + A_\theta \frac{V^1 V^2 V^3}{(V^0)^2 - (V^1)^2} \right) \partial\Theta, \end{aligned} \quad (74)$$

$$Z^2 = \partial V^2 + V^2 U_\theta \partial\Theta, \quad (75)$$

$$Z^3 = \partial V^3 - (V^3 U_\theta - \frac{e^{2U}}{s} V^2 A_\theta) \partial\Theta. \quad (76)$$

Then there is a matrix  $A = \{a_{lm}\}$ ,  $l, m = 0, 1, 2, 3$ , such that

$$\Omega := (\Psi, Z^1, Z^2, Z^3)^T$$

satisfies

$$\frac{d\Omega}{ds} = A\Omega, \quad (77)$$

and the matrix elements  $a_{lm} = a_{lm}(s, \Theta(s), V^k(s))$  are all uniformly bounded on  $[t_0, t_+]$ .

*Sketch of proof.* Once the ansatz (73)-(76) has been found this is only a lengthy calculation. To illustrate the type of calculations involved we show the easiest case, i.e. the  $Z^2$  term.

$$\begin{aligned}
\frac{dZ^2}{ds} &= \frac{d}{ds}(\partial V^2 + V^2 U_\theta \partial \Theta) \\
&= \partial \left( \frac{d}{ds} V^2 \right) + \frac{dV^2}{ds} U_\theta \partial \Theta \\
&\quad + V^2 (U_{t\theta} + U_{\theta\theta} \frac{d\Theta}{ds}) \partial \Theta + V^2 U_\theta \partial \left( \frac{d\Theta}{ds} \right).
\end{aligned} \tag{78}$$

Now we use (68) and (70) to substitute for  $d\Theta/ds$  and  $dV^2/ds$ . We find that the right hand side equals

$$\begin{aligned}
&\partial \left( -U_t V^2 - \sqrt{\alpha} U_\theta \frac{V^1 V^2}{V^0} \right) + \left( -U_t V^2 - \sqrt{\alpha} U_\theta \frac{V^1 V^2}{V^0} \right) U_\theta \partial \Theta \\
&+ V^2 (U_{t\theta} + U_{\theta\theta} \sqrt{\alpha} \frac{V^1}{V^0}) \partial \Theta + V^2 U_\theta \left( \frac{\alpha_\theta V^1}{2\sqrt{\alpha} V^0} \partial \Theta + \sqrt{\alpha} \partial \left( \frac{V^1}{V^0} \right) \right).
\end{aligned}$$

Taking the  $\partial$  derivative of the first term we find that all terms of second order derivatives and terms containing  $\alpha_\theta$  cancel. Next, since

$$-\sqrt{\alpha} U_\theta \partial \left( \frac{V^1 V^2}{V^0} \right) + \sqrt{\alpha} U_\theta V^2 \partial \left( \frac{V^1}{V^0} \right) = -\sqrt{\alpha} U_\theta \frac{V^1}{V^0} \partial V^2, \tag{79}$$

we are left with

$$\frac{dZ^2}{ds} = -(U_t V^2 + \sqrt{\alpha} U_\theta \frac{V^1 V^2}{V^0}) U_\theta \partial \Theta - (U_t + \sqrt{\alpha} U_\theta \frac{V^1}{V^0}) \partial V^2. \tag{80}$$

Finally we express this in terms of  $\Psi, Z^1, Z^2$  and  $Z^3$ . Here this is easy and we immediately get

$$\frac{dZ^2}{ds} = -(U_t + \sqrt{\alpha} U_\theta \frac{V^1}{V^0}) Z^2.$$

Clearly, the map  $(\partial \Theta, \partial V^k) \mapsto (\Psi, Z^k)$  is invertible so that this step is easy also in the other cases. It follows that the matrix elements  $a_{2m}$ ,  $m = 0, 1, 2, 3$ , are uniformly bounded on  $[t_0, t_+)$  (only  $a_{22}$  is nonzero here). The computations for the other terms are similar. For the  $Z^1$  term we point out that the evolution equations (36) and (37) should be invoked and that the matrix element  $a_{10}$  contains  $\eta_\theta$  and  $\alpha_\theta/2\alpha$ , but they combine and form  $\tilde{\eta}_\theta$ ,

and that derivatives of  $e^{-\eta}\Gamma$  and  $e^{-\eta}H_t$  appear. The latter terms are easily seen to be bounded in view of the auxiliary equations. For example, from (38) we get a bound on  $\partial_\theta(e^{-\eta}\Gamma)$ ,

$$\frac{\partial}{\partial\theta}(e^{-\eta}\Gamma) = -2e^{\eta+\tilde{\eta}-4U}J_2 - e^{\tilde{\eta}-\eta-4U}\Gamma\frac{\partial}{\partial\theta}(e^{-\tilde{\eta}+4U}).$$

The right hand side is bounded since  $U, U_\theta, \eta, \tilde{\eta}, \partial_\theta\tilde{\eta}$ , and  $\Gamma$  are all bounded as was shown in Step 1 and 2.

□

From the lemma it now immediately follows that  $|\Omega|$  is uniformly bounded on  $[t_0, t_+)$ . Moreover, since the system (73)-(76) is invertible with uniformly bounded coefficients we also have uniform bounds on  $|\partial\Theta|$  and  $|\partial V^k|$ ,  $k = 1, 2, 3$ . In view of the discussion at the beginning of this section we see that the distribution function  $f$  and the matter quantities  $\rho, J_k, S_{jk}$  and  $P_k$ , are all uniformly  $C^1$  bounded. From the constraint equation (34) we now obtain a uniform bound on  $\alpha_\theta$  by a simple Gronwall argument using as usual the identity  $\alpha e^{2(\eta-U)} = e^{2(\tilde{\eta}-U)}$ . Finally this yields a uniform bound on  $\eta_\theta$  since

$$\eta_\theta = \tilde{\eta}_\theta - \frac{\alpha_\theta}{2\alpha}$$

and  $\alpha$  stays uniformly bounded away from zero.

*Step 4.* (Bounds on second and higher order derivatives.)

It is now easy to obtain bounds on second order derivatives on  $U$  and  $A$  by using light cone arguments. We define  $X$  and  $Y$  by

$$X = \frac{1}{2}(U_{tt}^2 + \alpha U_{t\theta}^2) + \frac{e^{4U}}{8t^2}(A_{tt}^2 + \alpha A_{t\theta}^2), \quad (81)$$

$$Y = \sqrt{\alpha}U_{tt}U_{t\theta} + \frac{e^{4U}}{4t^2}A_{tt}A_{t\theta}, \quad (82)$$

and use the differentiated (with respect to  $t$ ) evolution equations for  $U$  and  $A$  to obtain equations similar to (55) and (56). In this case a straightforward light cone argument applies since we have control of the differentiated matter terms.  $U_{\theta\theta}$  and  $A_{\theta\theta}$  are then uniformly bounded in view of the evolution equations (36) and (37). Bounds on second order derivatives on  $f$  then follows from (77) by studying the equation for  $\partial\Omega$ . The only thing to notice is that  $\tilde{\eta}_{\theta\theta}$  is controlled by (33). It is clear that this reasoning can be continued to give uniform bounds on  $[t_0, t_+)$  for higher order derivatives as well.

□

## 5 Proofs of the main theorems

In this section the analytical and geometrical information obtained in previous sections is combined to obtain the main results of the paper.

*Proof of Theorem 2.* For a spacetime satisfying the hypotheses of the theorem we know from Section 2 that a conformal coordinate system can be introduced on a neighbourhood of the initial hypersurface  $S_0$  corresponding to the original data. The direct analogues of the results of Section 5 of [ARR] hold and can be proved by the same arguments. In fact the situation is slightly simpler since in the case of  $T^2$  symmetry there is an obvious choice of two Killing vectors while in [ARR] it was necessary to worry about choosing two from a total of three Killing vectors in an appropriate way. Cf. also [BCIM] where this type of argument was introduced for the first time. By these results it follows that the region where the solution exists can be extended to the past so as to include a Cauchy surface  $S_A$  of constant areal time. Moreover, either the conformal time coordinate extends to all negative values, or  $R$  tends to zero as the past boundary of the region covered by conformal coordinates is approached. In the first of these cases the region covered by the conformal time coordinate includes the entire past of the initial hypersurface in the maximal Cauchy development, as follows from the arguments of Section 5 of [ARR]. Also the past of  $S_A$  in that region admits a foliation by hypersurfaces of constant  $R$ . In that region we can transform to areal coordinates. For we can choose a new spatial coordinate  $\theta$  so that its coordinate lines in  $Q$  are orthogonal to that foliation. In the second case (where  $R$  tends to zero on the boundary of the region covered by conformal coordinates) the past of  $S_A$  is also covered by areal coordinates. It exhausts the past of  $S_A$  in the maximal Cauchy development, as will now be shown. If the spacetime could be extended to the past there would be a sequence of points tending to the boundary of the original spacetime in the extension. Along this sequence  $R$  would have to tend to zero. However the function  $R$  which is globally defined on the maximal Cauchy development must tend to a non-zero limit along the sequence approaching a point of the extension. Thus the existence of an extension leads to a contradiction. It follows that in both cases the entire past of  $S_0$  in the maximal Cauchy development is covered. As a consequence of the results of Section 4 the spacetime and the areal time coordinate can be extended so that the time coordinate covers the interval  $(R_0, \infty)$ . It can then be concluded by the argument at the end of Section 5 of [ARR] that the entire future of  $S_0$  in the maximal Cauchy development is covered.

□

*Proof of Theorem 1.* The mean curvature of the hypersurfaces of constant areal time is

$$\text{tr}k = -e^{-\eta+U}\alpha^{-1/2}(\eta_t - U_t + t^{-1}) \quad (83)$$

From the field equations it follows that

$$\eta_t - U_t + t^{-1} \geq tU_t^2 - U_t + t^{-1} = \frac{3}{4}tU_t^2 + t(\frac{1}{2}U_t - t^{-1})^2 \quad (84)$$

Hence there are Cauchy surfaces with everywhere negative mean curvature. Under these circumstances it was shown by Henkel [H] that the initial singularity is a crushing singularity and thus a neighbourhood of it can be foliated by CMC hypersurfaces. Given one CMC hypersurface the statement in Theorem 1 about the range of the CMC time coordinate follows from [R11]. It remains to see that the CMC foliation covers the entire future of the initial hypersurface. This can be proved by an argument used in the case of hyperbolic symmetry in [ARR] which will now be recalled. It is enough to show that if  $p$  is any point of the spacetime there is a compact CMC hypersurface which contains  $p$  in its past. Let  $S_1$  be the Cauchy surface of constant areal time passing through  $p$ . Equation (83) shows that the mean curvature of  $S_1$  is strictly negative. Hence it has a maximum value  $H_1 < 0$ . Let  $S_2$  be the compact CMC hypersurface with mean curvature  $H_1/2$ . Then the infimum of the mean curvature of  $S_2$  is greater than the supremum of the mean curvature of  $S_1$  and a standard argument [M] shows that  $S_2$  is strictly to the future of  $S_1$ . Hence  $p$  is in the past of  $S_2$ , as required.

□

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